

Negative Binomial States of Quantized Radiation Fields

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Abstract

We introduce the *negative binomial states* with negative binomial distribution as their photon number distribution. They reduce to the ordinary coherent states and Susskind-Glogower phase states in different limits. The ladder and displacement operator formalisms are found and they are essentially the Perelomov's $su(1,1)$ coherent states via its Holstein-Primakoff realisation. These states exhibit strong squeezing effect and they obey the super-Poissonian statistics. A method to generate these states is proposed.

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1 Introduction

Nonclassical states of the radiation fields, such as the number states, the coherent states and the phase states, play important roles in quantum optics and are extensively studied [1]. The binomial states (BS) introduced by Stoler *et al.* in 1985 [2], interpolate between the *most nonclassical* number states and the *most classical* coherent states, and reduce to them in two different limits. Some of their properties [2, 3, 4], methods of generation [2, 3, 5], as well as their interaction with atoms [6], have been investigated in the literature. The notion of BS was also generalised to the intermediate number-squeezed states [7, 8] and the number-phase states [9], the hypergeometric states [10], as well as their q -deformation [11].

The photon number distribution of BS is the binomial distribution in the probability theory [12, 13]. In this letter, we shall introduce and study *negative binomial states* (NBS) whose photon distribution is the *negative binomial distribution* (NBD) [12, 13]. Different from the BS, the NBS are the *intermediate phase-coherent states* in the sense that they reduce to the Susskind-Glogower (SG) phase states [14] and coherent states in two different limits (Sec.2). We also derive their ladder and displacement operator formalisms and find that they are essentially the Perelomov's $su(1,1)$ coherent states via its Holstein-Primakoff (HP) realizations (Sec.3). The NBS exhibit strong squeezing effects, but are not of sub-Poissonian statistics (Sec.4). A method to generate these NBS is proposed in Sec.5 and a summary with special emphasis on the comparison with BS is given in Sec.6.

2 Negative binomial states and their limits

We define the NBS as

$$|\eta e^{i\theta}; M\rangle^- = \sum_{n=0}^{\infty} \sqrt{B_n^-(\eta; M)} e^{in\theta} ||n\rangle, \quad (2.1)$$

in which $\{||n\rangle | n = 0, 1, \dots\}$ are the number states of the single-mode radiation field

$$[b, b^\dagger] = 1, \quad b||0\rangle = 0, \quad ||n\rangle = \frac{(b^\dagger)^n}{\sqrt{n!}} ||0\rangle, \quad (2.2)$$

M is a fixed positive integer, η^2 is the probability satisfying $0 < \eta^2 < 1$, and

$$B_n^-(\eta; M) = \binom{M+n-1}{n} \eta^{2n} (1-\eta^2)^M, \quad n = 0, 1, \dots \quad (2.3)$$

The $B_n^-(\eta; M)$ is called the NBD [12] since it can also be written as

$$B_n^-(\eta; M) = (1-\eta^2)^M \binom{-M}{n} (-\eta^2)^n, \quad n = 0, 1, \dots, \quad (2.4)$$

which has the similar form as the binomial distribution except for the two minuses and that n runs to infinity. The states (2.1) are referred to as the NBS since their photon distribution $|\langle n || \eta e^{i\theta}; M \rangle^-|^2 \equiv B_n^-(\eta; M)$ is the NBD.

The parameter θ ($0 \leq \theta < 2\pi$) has clear physical meaning: it reflects the time development of the NBS. This can be seen from $e^{-iHt} |\eta e^{i\theta}; M \rangle^- = |\eta e^{i(\theta-\omega t)}; M \rangle^-$, where $H = \omega(N + 1/2)$ is the Hamiltonian of the single mode radiation field.

As a probability distribution $B_n^-(\eta; M)$ satisfies [12]

$$\sum_{n=0}^{\infty} B_n^-(\eta; M) = 1, \quad (2.5)$$

which means that the NBS are normalized.

Let us consider two limiting cases:

(1). In the limit $\eta \rightarrow 0$, $B_n^-(\eta; M) \rightarrow \delta_{n0}$ and thus the NBS go to the vacuum state.

(2). When $\eta \rightarrow 0$, $M \rightarrow \infty$ with fixed finite $\eta^2 M = \alpha^2$, the NBD goes to the Poisson distribution $B_n^-(\eta; M) \rightarrow e^{-\alpha^2} \alpha^{2n} / n!$ [12]. Accordingly, the NBS degenerate to the ordinary coherent states.

We know that the BS degenerate to the number state in a certain limit [8]. However the NBS do not maintain this feature. Instead, the NBS tend to the SG phase states in a certain limit. To achieve this, let us consider the case $M = 1$. In this case, $|\eta e^{i\theta}; M \rangle^-$ is simplified as

$$|\eta e^{i\theta}; 1 \rangle^- = \sqrt{1 - \eta^2} \sum_{n=0}^{\infty} \eta^n e^{in\theta} |n \rangle. \quad (2.6)$$

The photon distribution $|\langle n || \eta e^{i\theta}; 1 \rangle^-|^2$ in this case is $(1 - \eta^2) \eta^{2n}$, the geometric distribution. For this reason we call $|\eta e^{i\theta}; 1 \rangle^-$ the geometric states. One can easily verify that the geometric state $|\eta e^{i\theta}; 1 \rangle^-$ is the eigenstate of the SG phase operator E^- [14] with the eigenvalue $\eta e^{i\theta}$

$$E^- = \sum_{n=0}^{\infty} |n \rangle \langle n + 1|, \quad (2.7)$$

which is related to the annihilation operator b through polar decomposition $b = E^- \sqrt{N}$. Now, multiplying a constant $1/\sqrt{2\pi(1 - \eta^2)}$ to $|\eta e^{i\theta}; 1 \rangle^-$ and then taking the limit $\eta \rightarrow 1$, we obtain the *phase states*

$$|\theta \rangle = \lim_{\eta \rightarrow 1} \frac{1}{\sqrt{2\pi(1 - \eta^2)}} |\eta e^{i\theta}; 1 \rangle^- = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{in\theta} |n \rangle, \quad E^- |\theta \rangle = e^{i\theta} |\theta \rangle. \quad (2.8)$$

The $|\theta \rangle$ states are non-normalisable, nonorthogonal, but resolve the identity

$$\int_{-\pi}^{\pi} d\theta |\theta \rangle \langle \theta| = 1. \quad (2.9)$$

The phase-state representation based on (2.9) is a useful calculational tool [15].

In this sense, we find that the NBS are the *intermediate phase-coherent states*. This is an important feature of the NBS.

3 Displacement and ladder operator formalisms

Let us recapitulate the displacement operator formalism of NBS [13] using the *identity method* developed in a previous paper [16]. To this end, let us rewrite NBS (2.1) ($\eta_C \equiv \eta e^{i\theta}$)

$$|\eta_C; M\rangle^- = (1 - |\eta_C|^2)^{\frac{M}{2}} \sum_{n=0}^{\infty} \frac{\sqrt{M(M+1) \cdots (M+n-1)}}{n!} (\eta_C)^n (b^\dagger)^n |0\rangle. \quad (3.1)$$

Then, by making use of the following identity

$$(b^\dagger g(N))^n |0\rangle = (b^\dagger)^n g(0)g(1) \cdots g(n-1) |0\rangle, \quad \text{with } g(N) \equiv \sqrt{M+N}, \quad N = b^\dagger b. \quad (3.2)$$

we can write Eq.(3.1) in the exponential form

$$\begin{aligned} |\eta_C; M\rangle^- &= (1 - |\eta_C|^2)^{\frac{M}{2}} \exp[\eta_C \mathcal{K}_+] |0\rangle, \\ \mathcal{K}_+ &= b^\dagger \sqrt{M+N} \equiv \sqrt{M-1+N} b^\dagger. \end{aligned} \quad (3.3)$$

It is interesting that \mathcal{K}_+ along with

$$\mathcal{K}_- = (\mathcal{K}_+)^{\dagger} \equiv \sqrt{M+N} b \equiv b \sqrt{M+N-1}, \quad \mathcal{K}_0 = \frac{M}{2} + N \quad (3.4)$$

generates the $su(1,1)$ algebra via its HP realization with the Bargmann index $M/2$. By making use of the disentangling theorem of $su(1,1)$ algebra [17] we arrive at the displacement operator formalism of NBS

$$|\eta e^{i\theta}; M\rangle^- = \exp[\zeta_C \mathcal{K}_+ - \zeta_C^* \mathcal{K}_-] |0\rangle, \quad \zeta_C = e^{i\theta} \operatorname{arctanh} \eta. \quad (3.5)$$

Eq.(3.5) is nothing but the Perelomov's coherent state of $su(1,1)$ via its HP realisation.

The Perelomov's coherent states admit the ladder operator form [18]. To see this, we differentiate both (3.3) and (3.5) with respect to $|\zeta|$ and equate the results. We have

$$[e^{-i\theta} \mathcal{K}_- - \eta^2 e^{i\theta} \mathcal{K}_+] |\eta e^{i\theta}; M\rangle^- = M \eta |\eta e^{i\theta}; M\rangle^-. \quad (3.6)$$

This ladder operator form is obviously compatible with the limit results. In fact, in the limits $\eta \rightarrow 0$ and $\eta \rightarrow 0$, $M \rightarrow \infty$ with $M\eta^2 = \alpha^2$, Eq.(3.6) reduces to

$$b|0; M\rangle^- = 0, \quad b|0; \infty\rangle^- = \alpha e^{i\theta} |0; \infty\rangle^-, \quad (3.7)$$

which mean $|0; M\rangle^-$ and $|0; \infty\rangle^-$ are the vacuum and coherent states respectively.

Finally we point out that the NBS can also be regarded as the *density dependent annihilation operator* coherent states, namely,

$$E_M^- |\eta e^{i\theta}; M\rangle^- = e^{i\theta} \eta |\eta e^{i\theta}; M\rangle^-, \quad E_M^- \equiv b \frac{1}{\sqrt{N+M-1}} \quad (3.8)$$

when $M \geq 2$. The operator E_1^- is not well-defined in the vacuum state. If we require

$$E_1^- ||0\rangle = 0, \quad (3.9)$$

then it is obvious that $E_1^- = E^-$. This connection of SG phase operator with Perelomov's coherent states with Bargmann index $1/2$ was well-known.

4 Nonclassical effects

4.1 Photon statistics

Let us first examine if the NBS is of sub-Poissonian statistics. Using Eq. (2.5), it is easy to calculate the averages $\langle N \rangle$, $\langle N^2 \rangle$ and the fluctuation $\langle \Delta N^2 \rangle$ of the photon number

$$\langle N \rangle = \frac{M\eta^2}{1-\eta^2}, \quad \langle N^2 \rangle = \frac{M\eta^2 + M^2\eta^4}{(1-\eta^2)^2}, \quad \langle \Delta N^2 \rangle = \frac{M\eta^2}{(1-\eta^2)^2}. \quad (4.1)$$

Then Mandel's Q -factor characterising sub-Poissonian (if $Q < 0$) distribution is obtained as

$$Q = \frac{\langle \Delta N^2 \rangle - \langle N \rangle}{\langle N \rangle} = \frac{\eta^2}{1-\eta^2} > 0. \quad (4.2)$$

So the field in NBS is *super-Poissonian*, not sub-Poissonian, except for the (vacuum and coherent state) limit $\eta \rightarrow 0$ ($Q \rightarrow 0$, Poissonian statistics).

Since the occurrence of antibunching and sub-Poissonian are concomitant for single and time independent field, as in the present case, the field in NBS is bunching, not antibunching. In fact, from the following second-order correlation function $g^2(0)$

$$g^2(0) = \frac{\langle b^\dagger b^\dagger b b \rangle}{\langle b^\dagger b \rangle^2} = 1 + \frac{1}{M} > 1, \quad (4.3)$$

we can arrive at the same conclusion.

4.2 Squeezing effect

Let us evaluate the variances $\langle \Delta x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$ and $\langle \Delta p^2 \rangle = \langle p^2 \rangle - \langle p \rangle^2$ of the quadrature operators x (coordinate) and p (momentum) defined by

$$x = \frac{1}{\sqrt{2}}(b^\dagger + b), \quad p = \frac{i}{\sqrt{2}}(b^\dagger - b). \quad (4.4)$$

From the following relation

$$b^k |\eta e^{i\theta}; M\rangle^- = \left(\frac{\eta e^{i\theta}}{\sqrt{1-\eta^2}} \right)^k \sqrt{M(M+1)\cdots(M+k-1)} |\eta e^{i\theta}; M+k\rangle^-, \quad (4.5)$$

we have

$$\begin{aligned} \langle \Delta p^2 \rangle &= \frac{1}{2} - \frac{\eta^2(1-\eta^2)^M}{(M-1)!} \sum_{n=0}^{\infty} \eta^{2n} \frac{(M+n-1)!}{n!} \sqrt{M+n} \left(\cos(2\theta) \sqrt{M+n+1} - \sqrt{M+n} \right) \\ &\quad - 2 \sin^2 \theta \eta^2 (1-\eta^2)^{2M} \left[\sum_{n=0}^{\infty} \binom{M+n-1}{n} \eta^{2n} \sqrt{M+n} \right]^2, \end{aligned} \quad (4.6)$$

$$\begin{aligned} \langle \Delta x^2 \rangle &= \frac{1}{2} + \frac{\eta^2 M}{1-\eta^2} + \frac{\cos(2\theta) \eta^2 (1-\eta^2)^M}{(M-1)!} \sum_{n=0}^{\infty} \eta^{2n} \frac{(M+n-1)!}{n!} \sqrt{(M+n+1)(M+n)} \\ &\quad - 2 \cos^2 \theta \eta^2 (1-\eta^2)^{2M} \left[\sum_{n=0}^{\infty} \binom{M+n-1}{n} \eta^{2n} \sqrt{M+n} \right]^2. \end{aligned} \quad (4.7)$$

In the derivation of Eq.(4.6), we have used the identity

$$\begin{aligned} \frac{M\eta^2}{1-\eta^2} &= \frac{\eta^2(1-\eta^2)^M}{(M-1)!} \frac{M!}{(1-\eta^2)^{M+1}} = \frac{\eta^2(1-\eta^2)^M}{(M-1)!} \sum_{n=0}^{\infty} \eta^{2n} \frac{(M+n)!}{n!} \\ &= \frac{\eta^2(1-\eta^2)^M}{(M-1)!} \sum_{n=0}^{\infty} \eta^{2n} \frac{(M+n-1)!}{n!} \sqrt{M+n} \sqrt{M+n}. \end{aligned} \quad (4.8)$$

Let us analyse $\langle \Delta p^2 \rangle$ in more detail. First consider the case $\theta = 0$ (the initial time). In this case Eq.(4.6) is simplified as

$$\langle \Delta p^2 \rangle = \frac{1}{2} - \frac{\eta^2(1-\eta^2)^M}{(M-1)!} \sum_{n=0}^{\infty} \eta^{2n} \frac{(M+n-1)!}{n!} \sqrt{M+n} \left(\sqrt{M+n+1} - \sqrt{M+n} \right). \quad (4.9)$$

Since every term in the infinite series in Eq.(4.9) is positive, the infinite sum is positive. So we always have $\langle \Delta p^2 \rangle < 1/2$. This means that the quadrature p is squeezed. Fig.1 shows how $P \equiv \langle \Delta p^2 \rangle$ depends on η and M . We have chosen $0 \leq \eta^2 \leq 0.99$. Those plots show that:

(1). Dependence on η^2 . It is found that the variance $\langle \Delta p^2 \rangle$ is an increasing function of η^2 , namely, the larger η^2 , the stronger the squeezing effect. Note that $\langle \Delta p^2 \rangle > 0$ since $\langle \Delta p^2 \rangle = \langle p^2 \rangle$ and p^2 is a positive definite hermitian operator.

(2). Dependence on M . We have chosen $M=1, 5, 50$. The squeezing of p for larger M is stronger than that for small M . When η^2 is small (close to 0) or large (close to 1), the difference is very small. While when η^2 is around 1/2, the difference is larger. However, squeezing is not so sensitive to M . The case $M=5$ and $M=50$ are almost same as showed in Fig.1.

Furthermore, due to the uncertainty relation $\langle \Delta x^2 \rangle \langle \Delta p^2 \rangle \geq 1/4$, the variance $\langle \Delta x^2 \rangle > 1/2$ when $\theta = 0$. So the quadrature x is not squeezed.

When $\theta \neq 0$, it is easy to see that $\langle \Delta p^2 \rangle$ is a π -periodic function with respect to θ and symmetric with respect to $\theta = \pi/2$. To investigate the effect of θ to $\langle \Delta p^2 \rangle$, we plot the $\langle \Delta p^2 \rangle$ as a function of θ for different η^2 values in Fig.2 (we choose $M = 1$ for simplicity). We find that (1) $\langle \Delta p^2 \rangle$ becomes larger when θ creasing. It first reaches 1/2 and then reaches the maximal value when $\theta = \pi/2$. Then it symmetrically decreases until $\theta = \pi$. In some region around $\theta = \pi/2$ (dependent on η^2), the squeezing disappears. (2) Small η^2 violates the squeezing slightly while large η^2 destroys the squeezing strongly.

5 Generation of NBS

The displacement operator formalism suggests that the NBS can be generated by the non-degenerate parameter amplifier described by the Hamiltonian [19]

$$H = H_0 + \chi i(a_1^\dagger a_2^\dagger e^{-2i\omega t} - a_1 a_2 e^{2i\omega t}), \quad H_0 = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2, \quad (5.1)$$

where a_1 and ω_1 (a_2 and ω_2) are the annihilation operator and frequency for the signal (idler) mode. Frequencies ω_1 and ω_2 sum to the pump frequency, $2\omega = \omega_1 + \omega_2$. The coupling constant χ is proportional to the second-order susceptibility of the medium and to the amplitude of the pump. The unitary time evolution operator in the interaction picture is

$$U(t) = e^{iH_0 t} e^{\chi t(a_1^\dagger a_2^\dagger - a_1 a_2)} e^{-iH_0 t}. \quad (5.2)$$

Suppose that the system is initially prepared in the state $|0, M\rangle \equiv ||0\rangle$. Since the photons are created or annihilated in pairs, we can restrict ourselves in the subspace

$$||n\rangle \equiv |n, n + M\rangle, \quad n = 0, 1, 2, \dots, \quad (5.3)$$

which is isomorphic to the single-mode Fock space. Then at any time t the system is in

$$U(t)||0\rangle = \sum_{n=0}^{\infty} \left[\binom{n+M-1}{n} (1 - \tanh^2 \chi t)^M (\tanh \chi t)^{2n} \right]^{\frac{1}{2}} e^{i2\omega n} ||n\rangle. \quad (5.4)$$

Identifying $\eta = \tanh \chi t$ and $\theta = 2\omega t$, we obtain the NBS.

6 Conclusion

In this letter we have introduced the negative binomial states and studied their nonclassical properties. The following table shows some properties of NBS with the special emphasis on the comparison with the binomial states:

	NBS	BS
Limit States	Coherent and SG phase states	Coherent and number states
Related algebra	$su(1, 1)$	$su(2)$
Essence	Perelomov's coherent states	Atomic coherent states
Sub-Poissonian	No	Unconditional
Antibunching	No	Unconditional
Squeezing effect ($\theta = 0$)	Unconditional for p , no squeezing for x	Limited ranges of parameters for x , no squeezing for p

We see that the nonclassical properties of the NBS and BS are complementary.

As further work we shall generalize the notations of NBS in this letter to the *negative multinomial states* with negative multinomial distribution as their photon distribution. This generalisation concerns the multi-mode radiation field and should exhibit some more fruitful nonclassical properties like correlation between different modes. It is also a good challenges to study the interaction of radiation field in the NBS with the atoms.

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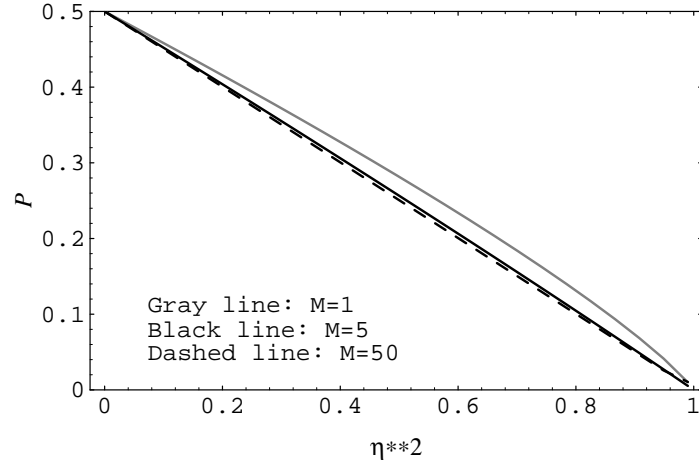


Figure 1: Variance $P \equiv \langle \Delta p^2 \rangle$ as a function of $\eta^2 (\equiv \eta^{**2})$ for $\theta = 0$ (initial time) and $M=1$ (gray line), 5 (black line) and 50 (dashed line).

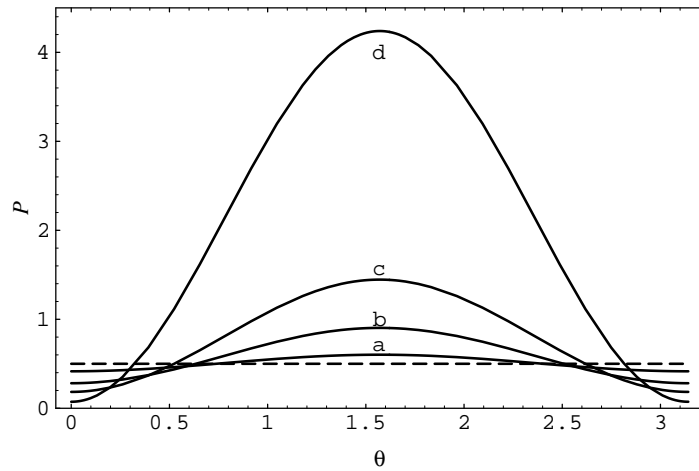


Figure 2: Variance $P \equiv \langle \Delta p^2 \rangle$ as a function of θ for different η^2 , 0.2 (a), 0.5 (b), 0.7 (c) and 0.9 (d). In all the cases $M = 1$. The dashed line corresponds to $\eta = 0$ or $P = 0.5$.